On Wednesday, Nov 3rd in class I was discussing different ways of saying things like “There is exactly one painter”. Letting “P(x)” stand for “x is a painter” we can count in the following way:

There is at least one painter: $\exists x\, P(x)$
There is at least two painters: $\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$
There is at most one painter = It is false that there are at least two painters: $\neg\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$

By DeMorgan’s for quantifiers, this is equivalent to:
$\forall x \forall y((P(x) \wedge P(y)) \rightarrow x = y)$

There is exactly one painter = there is at least one and at most one:
$\exists x\, P(x) \wedge \neg\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$ which is equivalent to
$\exists x\, P(x) \wedge \forall x \forall y((P(x) \wedge P(y)) \rightarrow x = y)$ which is equivalent to
$\exists x(P(x) \wedge \forall y(P(y) \rightarrow x = y))$

Now in class I asked how we could reduce this to one quantifier and I said that the following works:
$\forall y(P(y) \leftrightarrow x = y)$

This is not correct and this is not even a sentence.

There are at least there interesting kinds of sentences we might think about here.
$\exists x \forall y(P(y) \leftrightarrow x = y)$
$\forall x \forall y(P(y) \leftrightarrow x = y)$
$\forall y(P(y) \leftrightarrow a = y)$

The first one says that there is exactly one P.
The second one says that there is exactly one thing and it is a P.
The third one says that there is exactly one P and it is ‘a’ is that P.

While this third sentence does imply that there is exactly one painter, it is not equivalent to it. “There is exactly one painter” is consistent with ‘a’ referring to a non-painter while ‘b’ is the painter. In fact there is no sentence that has only one quantifier that is equivalent to ‘exactly one P’.